Introduction to Volterra series and applications to physical audio signal processing

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Vito Volterra [1860 (Ancona) - 1940 (Roma)] (source: wikipedia)

Vito Volterra was an **Italian mathematician and physicist**. He is known for his contributions to **mathematical biology and integral equations**. He joined the opposition to the Fascist regime (1922). Because of his political philosophy, he also refused to take a mandatory oath of loyalty (1931). He lived largely abroad, returning to Rome just before his death.

Royal Society (1910) - Royal Society of Edinburgh (1913)
A moon crater is named after him

Outline

A. Introduction & Motivation

B. **Volterra series:**

- B1. Definitions and basic properties
- B2. Interconnection laws

C. **TUTORIAL: How to ...**

- C1. derive the Volterra kernels of a given nonlinear system ?
- C2. simulate the system using these kernels ?

[Short pause (≈ 5min): first questions]

D. **Physical/Audio applications:**

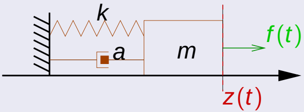
- D1. Nonlinear propagation of a travelling wave (brassy effect)
- D2. A damped nonlinear string

E. **Recent results** on computable convergence domains

F. Conclusion

A1. AN EXAMPLE: Mass-Spring-Damper system

Problem (at rest for $t < 0$)



$$m z''(t) + a z'(t) + k z(t) = f(t)$$

Find the trajectory $z(t)$ with respect to the force $f(t)$

System with input (u) / Output (y)

$u := f \longrightarrow$ Closed-form solution ? $\longrightarrow y := z$

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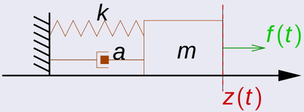

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State-space representation: $x(t) = [z(t), z'(t)]^T$, $x(0) = [0, 0]^T$

$$\underbrace{\begin{bmatrix} z'(t) \\ z''(t) \end{bmatrix}}_{x'(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -a/m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} z(t) \\ z'(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} u(t)$$

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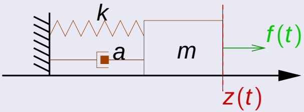

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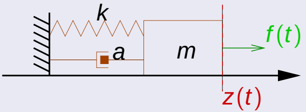

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$$y(t) = \int_0^t h(\tau) u(t-\tau) \mathrm{d}\tau$$

Convolution(/filtering) with the impulse reponse

$h(\tau) = C \, e^{A\tau} B \, 1_{\tau \geq 0}$

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A2. AN EXAMPLE: in the LAPLACE domain

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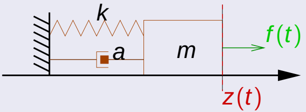

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$$u := f \longrightarrow \boxed{\text{Closed-form solution ?}} \longrightarrow y := z$$

$$Y(s) = H(s)\, U(s)$$

Transfer function (/filter)

$$H(s) = C\,(s I_2 - A)^{-1} B$$

State-space representation: $x(t) = [z(t), z'(t)]^T$, $x(0) = [0, 0]^T$

$$\begin{aligned} \text{Eq.}: \; s X(s) &= A X(s) + B U(s) \\ \text{Sol.}: \; X(s) &= (s I_2 - A)^{-1} B U(s) \end{aligned}$$

$$\begin{aligned} Y(s) &= [1, 0] \; X(s) \\ &= C \; X(s) \end{aligned}$$

A3. What about nonlinear systems ?

Input/Output nonlinear differential system (state x)

$$\begin{aligned} x'(t) &= F\big(x(t), u(t)\big) \\ y(t) &= G\big(x(t), u(t)\big) \end{aligned}$$

$u \longrightarrow$ Closed-form solution? $\longrightarrow y$

In general, NO ! But...

Linear case

$$F(x,u) = Ax + Bu$$
$$G(x,u) = Cx + Du$$

I/O relation: linear filter

kernel: $h(t) = Ce^{At}B + D$

trsf. fct: $H(s) = C(sI + A)^{-1}B + D$

Interests: frqcy domain analysis, simulation, etc.

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F, G: power series expansions *around equilibrium point 0 (nonzero coeff. at order 1)*

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I/O relation: Volterra series

Interests: idem!

A4. From a qualitative point of view...

A few comparisons:

Case	closed-form sol. w.r.t. input	distortions	self-oscillations bifucations, chaos
General	no	yes	yes
Volterra	yes	yes	no
Linear	yes	no	no

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Interest of Volterra series:

- **Natural distortions** for high amplitudes
- Musical acoustics: **fortissimo dynamics**
- Possible extensions to **partial differential equations**

What is the idea ? (regular perturbation method)

For a Weakly Nonlinear System ...

$$x'(t) = F\big(x(t), u(t)\big) \qquad F(X,U) = \sum_{m,n} \frac{D_{m,n}F(0,0)}{m!n!}(X,\ldots,X,U,\ldots,U)$$

$$y(t) = G\big(x(t), u(t)\big) \qquad G(X,U) = \sum_{m,n} \frac{D_{m,n}G(0,0)}{m!n!}(X,\ldots,X,U,\ldots,U)$$

Consider the input as a small perturbation of the system.

Mark it with $\eta \in (0,1)$: $u(t) = \eta v(t)$

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(i) Introduce $x(t) = \sum_n \eta^n x_n(t)$ and $y(t) = \sum_n \eta^n y_n(t)$

(ii) Inject these series expansions in the system equations

(iii) Sort equations w.r.t. η^n

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(iv) Solution: Each x_n is a **multiple convolution** of n repeated versions of the input and a computable multivariate kernel

$\longrightarrow$ **Volterra kernel**

B. Volterra series

VOLTERRA SERIES:

B1. **Definitions and basic properties**

B2. **Interconnection laws**

B1. Volterra series

Part B1. Definitions and basic properties:

1. Definition and particular cases
2. Laplace/Fourier domain and analogies with linear systems
3. Case of periodic signals, distortion coefficients
4. Remark on the non uniqueness of kernels

B1.1. Volterra series: definition and particular cases

Definition

A system $\xrightarrow{u} \boxed{\{h_n\}} \xrightarrow{y}$ is defined by the Volterra series $\{h_n\}_{n\geq 1}$ if

$$y(t) = \underbrace{\sum_{n=1}^{+\infty}}_{\text{Sum}} \underbrace{\int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n)\, u(t-\tau_1) \ldots u(t-\tau_n)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n}_{\text{of multiple convolutions}}$$

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$$= \int_{\mathbb{R}} h_1(\tau_1) u(t-\tau_1) \, \mathrm{d}\tau_1$$

Example

- Linear filters: $h_n = 0, \quad \text{if } n \geq 2.$

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- Linear filters: $h_n = 0, \quad \text{if } n \geq 2.$
- Memoryless fct: $h_n(\tau_1, \dots, \tau_n) = \alpha_n \, \delta(\tau_1, \dots, \tau_n),$ (δ: Dirac).

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A definition is also available for time-varying systems:

$$y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} g_n(t,\tau_1,\ldots,\tau_n)\, u(\tau_1)\ldots u(\tau_n)\, d\tau_1\ldots d\tau_n$$

(not presented in this tutorial)

B1.2. Analogies with linear systems & Laplace/Fourier domain

A system is causal if

$$\tau < 0 \quad \Rightarrow \quad h(\tau) = 0 \qquad \text{(linear)}$$
$$\tau_k < 0 \quad \Rightarrow \quad h_n(\tau_1, \ldots, \tau_k, \ldots, \tau_n) = 0 \qquad \text{(Volterra)}$$

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Laplace domain (or Fourier domain with $s = 2i\pi f$)

Transfer function: $H(s) = \int_{\mathbb{R}} h(\tau) e^{-s\tau} \, d\tau$ (lin.)

Transfer kernel: $H_n(s_{1:n}) = \int_{\mathbb{R}^n} h_n(\tau_{1:n}) e^{-(s_1\tau_1 + \ldots + s_n\tau_n)} \, d\tau_1 \ldots d\tau_n$ (Volt.)

denoting $(s_{1:n}) = (s_1, \ldots, s_n)$ and $(\tau_{1:n}) = (\tau_1, \ldots, \tau_n)$.

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Causal stable system: NO poles (and NO singularities)

of $H(s)$ for $\Re e(s) > 0$ (linear)

of $H_n(s_{1:n})$ for $\Re e(s_k) > 0$ (Volterra)

B1.3. Periodic signals and distortion coefficient

Analytic input signal $u(t) = a\,e^{i\omega t}$

$$u(t) = a\,e^{i\omega t} \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t) = \sum_{n=1}^{+\infty} a^n H_n(i\omega, \dots, i\omega)\, e^{in\omega t}$$

Periodic input signals / Fourier series

$$u(t) = \sum_{k=-\infty}^{+\infty} u_k\, e^{ik\omega t} \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} y_k\, e^{ik\omega t}$$

with $y_k = \sum_{n=1}^{+\infty} \sum_{\substack{k_1,\dots,k_n=-\infty \\ k_1+\dots+k_n=k}}^{+\infty} u_{k_1} \dots u_{k_n} H_n(ik_1\omega, \dots, ik_n\omega)$

Distortion coefficient for $u(t) = a\cos(\omega t)$

$D(a, \omega) = \sum_{n=2}^{+\infty} |y_n|^2 / |y_1|^2$: closed-form solution w.r.t. a, ω, H_n.

B1.4. Non-uniqueness of Volterra kernels

Remark:

Permuting variables τ_k in $y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n) u(t-\tau_1) \ldots u(t-\tau_n) \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n$ leaves the output y unchanged.

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Example

$h_2(\tau_1, \tau_2)$, $h_2(\tau_2, \tau_1)$, but also $\alpha h_2(\tau_1, \tau_2) + (1-\alpha) h_2(\tau_2, \tau_1)$ define the same Input-Ouput system.

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for all permutations π, $h_n(\tau_{\pi(1)}, \dots, \tau_{\pi(n)}) = h_n(\tau_1, \dots, \tau_n)$

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Symmetrical versions of Volterra kernels $SYM(h_n)$ are unique

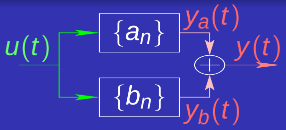
$$\mathrm{SYM}[h_n](\tau_1, \ldots, \tau_n) = \frac{1}{n!} \sum_\pi h_n(\tau_{\pi(1)}, \ldots, \tau_{\pi(n)})$$

Other unique versions (triangular kernels, regular kernels) are also available.

B2. Volterra series

Part B2. Interconnection laws of Volterra series:

1. Sum
2. Product
3. Cascade



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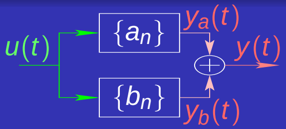

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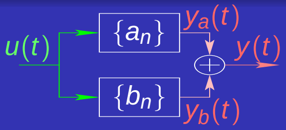

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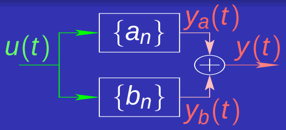
$$\begin{aligned} y(t) &= \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} a_n(\tau_{1:n}) u(t-\tau_1) \dots u(t-\tau_n)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n \\ &\quad + \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} b_n(\tau_{1:n}) u(t-\tau_1) \dots u(t-\tau_n)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n \\ &= \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} \left[a_n(\tau_{1:n}) + b_n(\tau_{1:n})\right] u(t-\tau_1) \dots u(t-\tau_n)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n \end{aligned}$$

B2.1. Interconnection laws: SUM



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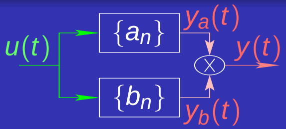
Result: Equivalent kernels c_n

$$c_n(\tau_{1:n}) = a_n(\tau_{1:n}) + b_n(\tau_{1:n})$$

Laplace T.: $$C_n(s_{1:n}) = A_n(s_{1:n}) + B_n(s_{1:n})$$

B2.2. PRODUCT



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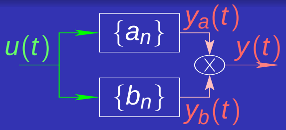

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\end{aligned}
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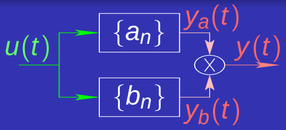

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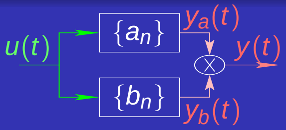

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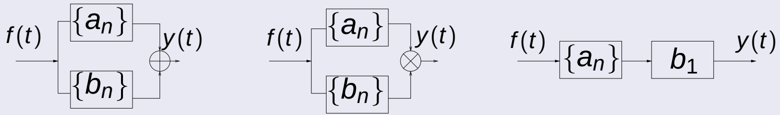
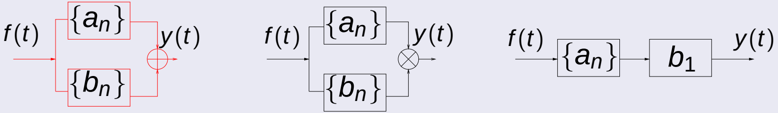
B2.3. Interconnection laws: Sum, Product and Cascade



Equivalent transfer kernels C_n

B2.3. Interconnection laws: Sum, Product and Cascade



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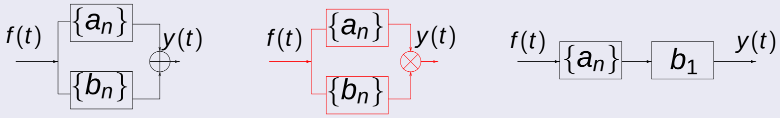

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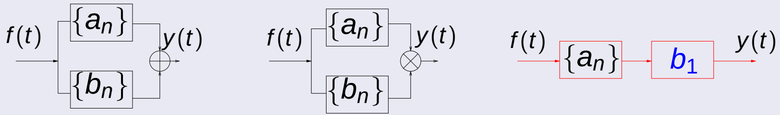

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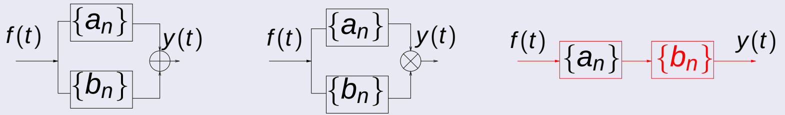

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Cascade: $$C_n(s_{1:n}) = \sum_{m=1}^{n} \sum_{q_1+\ldots+q_m=n} A_{q_1}(s_{1:q_1}) \ldots A_{q_m}(s_{q_1+\ldots+q_{m-1}+1:n}) . B_m\left(\widehat{s_{1:q_1}}, \ldots\ldots, \widehat{s_{q_1+\ldots+q_{m-1}+1:n}}\right)$$

(general case)

Part B: In summary:

A Volterra series ...

- catches distortions (memory combined with nonlinearities)
- sorts the nonlinear responses w.r.t. the degree n of homogenous contributions of u
- generalizes the convolution principle
- can be described by transfer kernels in the frequency domain (as filters).
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Moreover,

Interconnections of Volterra series (sum, product, cascade) define a Volterra series with computable kernels.

Part C: Tutorial

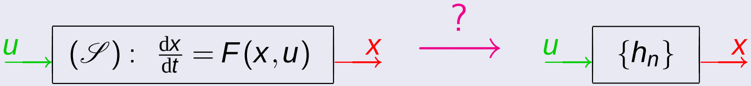
TUTORIAL: How to ...

C1. derive the Volterra kernels of a given nonlinear system ?

C2. simulate the system using these kernels ?

C1.1. How to derive the Volterra kernels of a system $\mathscr{S}$?

Goal: Find the Volterra kernels $\{h_n\}$ of $(\mathscr{S})$ where



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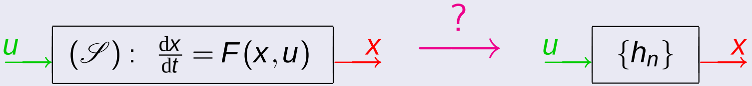

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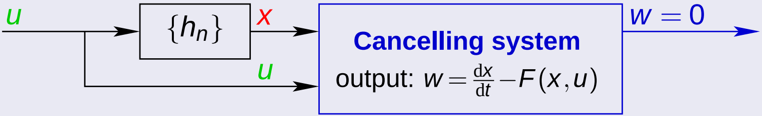
Here: Introduce the **"cancelling system"** of $\mathscr{S}$



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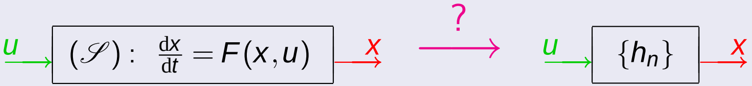

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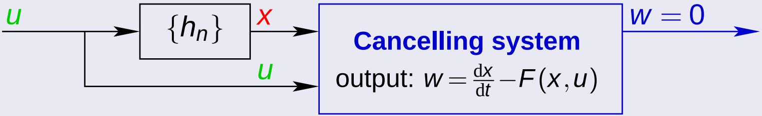


Principle: this cascade defines the **null system** $\xrightarrow{u} \boxed{\{z_n = 0\}} \xrightarrow{w} = 0$

Interconnection laws gives the equations satisfied by $\{h_n\}$.

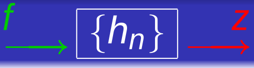
C1.2. Example: nonlinear spring

$f \longrightarrow \boxed{\{h_n\}} \longrightarrow z$

Equation (at rest before t=0)

$$mz'' + az' + k_1 z + k_2 [z]^2 = f$$

C1.2. Example: nonlinear spring



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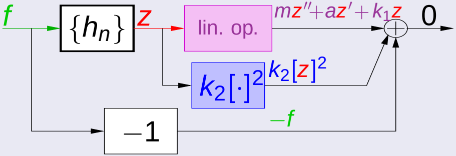
Cancelling system (block diagram)



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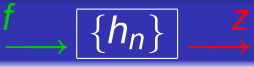

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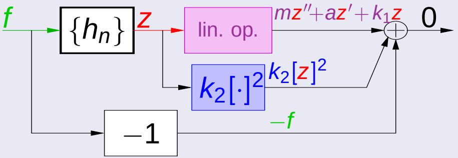

Elementary blocks → equivalent transfer kernels

$m\frac{d^2}{dt^2} + a\frac{d}{dt} + k_1$ → $Q_1(s) = ms^2 + as + k_1$, $Q_n = 0$ si $n \geq 2$

$k_2[\cdot]^2$ → interconnection "product" and $\times k_2$

-1 → $-\delta_{1,n} = -1$ if $n = 1$ and $-\delta_{1,n} = 0$ otherwise

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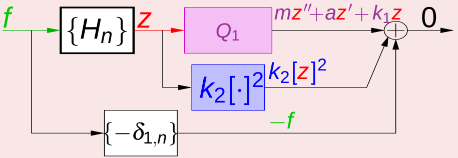

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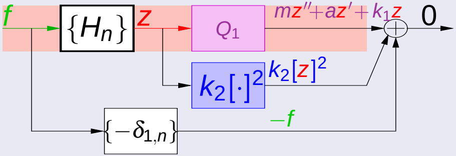

Kernel of order *n* of the *cancelling system*

$$H_n(s_{1:n})Q_1(\widehat{s_{1:n}})$$

$$+ \; k_2 \sum_{p=1}^{n-1} H_p(s_{1:p}) H_{n-p}(s_{p+1:n})$$

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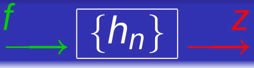

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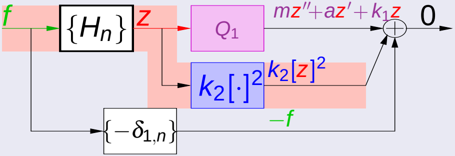

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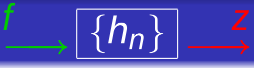

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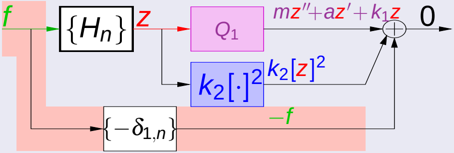

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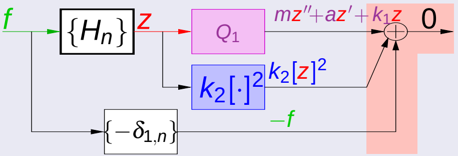

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$$+ \quad -\delta_{1,n} \qquad = 0 \longrightarrow \text{linear eq. w.r.t. } H_n.$$

C1.3. Kernels $\{H_n\}$ of the system $\xrightarrow{f} \boxed{\{h_n\}} \xrightarrow{z}$

General solution: recursive algebraic equation ($n \geq 1$)

$$H_n(s_{1:n}) = \frac{\delta_{1,n} - k_2 \overbrace{\sum_{p=1}^{n-1} H_p(s_{1:p}) H_{n-p}(s_{p+1:n})}^{\text{orders} < n}}{Q_1(\widehat{s_{1:n}})}$$

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First transfer kernels ($n = 1, 2, 3, \text{etc}$)

$H_1(s_1) = 1/Q_1(s_1)$, (second order AR filter)

$H_2(s_{1:2}) = -k_2\, H_1(s_1)\, H_1(s_2)\, H_1(\widehat{s_{1:2}})$,

$H_3(s_{1:3}) = -k_2 \left[H_2(s_{1:2})\, H_1(s_3) + H_1(s_1)\, H_2(s_{2:3})\right] H_1(\widehat{s_{1:3}})$,

etc.

Part C: Tutorial

TUTORIAL: How to ...

C2. simulate the system using these kernels ?

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Several methods are available... (realization theory in [Rugh])

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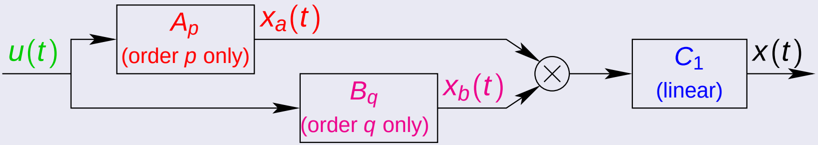

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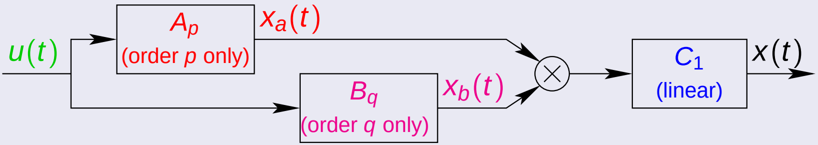

Equivalent system (using “product” & “cascade”)

The Volterra transfer kernels of this system are all zero except

$$H_{p+q}(s_{1:p+q}) = A_p(s_{1:p})\, B_q(s_{p+1:p+q})\, C_1(\widehat{s_{1:p+q}})$$

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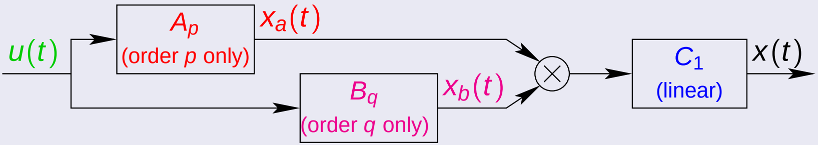

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Method:

Recursively build a realization as a **sum of such systems**.

C2.2. Example: nonlinear spring

Transfer kernels (recalls of C1.3)

$H_1(s_1) = 1/Q_1(s_1)$, (AR filter)

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Transfer kernels (recalls of C1.3)

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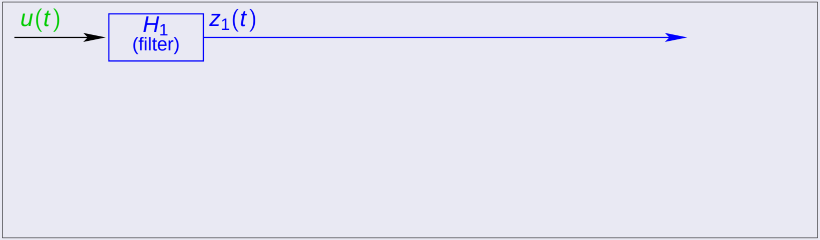
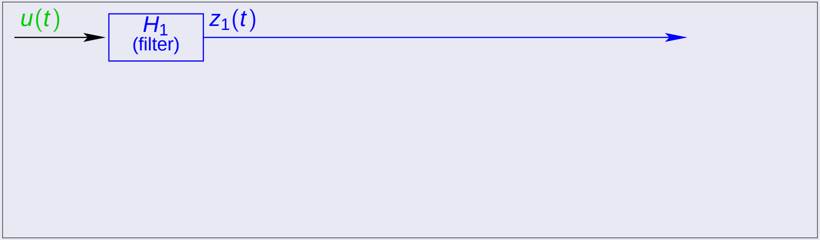


C2.2. Example: nonlinear spring

Transfer kernels (recalls of C1.3)

$$H_2(s_{1:2}) = -k_2\, H_1(s_1)\, H_1(s_2)\, H_1(\widehat{s_{1:2}})$$



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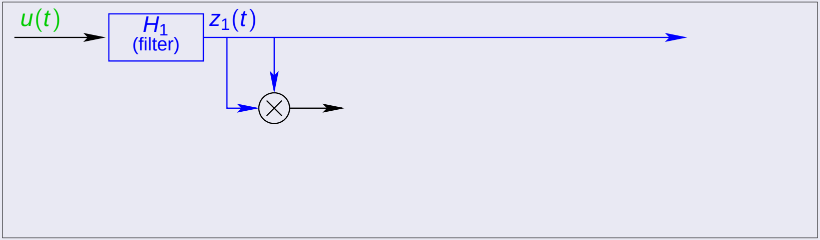

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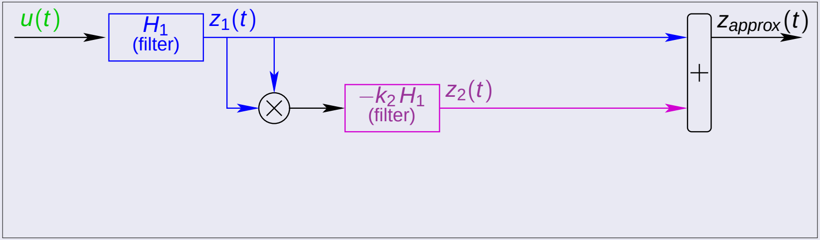

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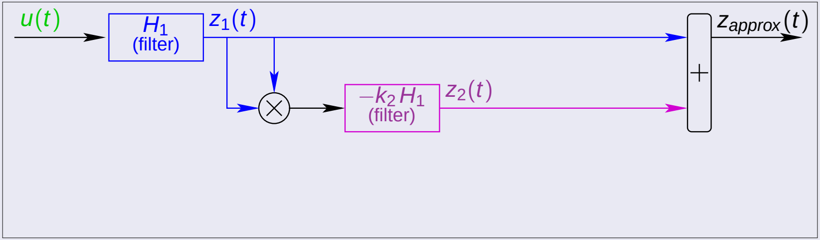
Transfer kernels (recalls of C1.3)

$$H_3(s_{1:3}) = -k_2 \left[H_2(s_{1:2})\, H_1(s_3) + H_1(s_1)\, H_2(s_{2:3}) \right] H_1(\widehat{s_{1:3}})$$



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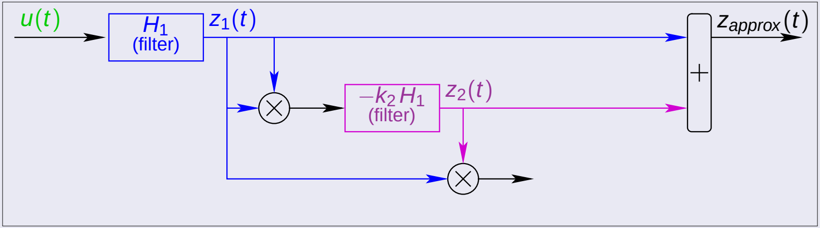

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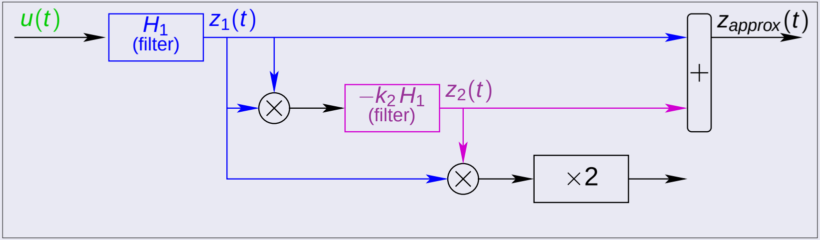

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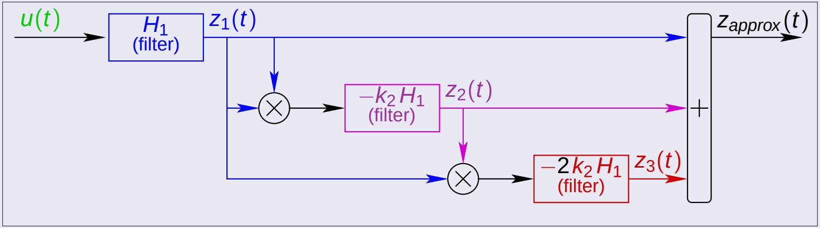

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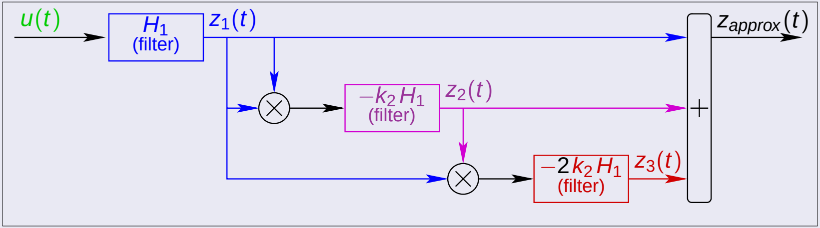

RESULT: The system is composed of **sums**, **products** and **linear filters** ($\longrightarrow$ just implement digital versions!)

Rk: if linear filters are stable, the system is stable!

C2.3. Aliasing rejection in simulations

Product of signals

signal	$a(t)$	$b(t)$	$c(t) = a(t)\,b(t)$
frequency range	$(-f_a, f_a)$	$(-f_b, f_b)$	$(-f_a - f_b, f_a + f_b)$

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Aliasing rejection

(Global sol.) Oversample the input/Downsample the output: factor N for a VS truncated at order N.

(Local sol.) Idem with a factor 2 for each product of two signals.

Part C: In summary...

Derivation of the transfer kernels

1. Use the cancelling system and interconnection laws...
2. ... to transform the weakly nonlinear problem into a infinite sequence of solvable linear equations

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Simulation in practice

1. Truncate the series to catch the first distortions
2. Decompose the kernels into sums of elementary systems
3. Build the corresponding structure composed of **linear filters**, **sums** and **products** of signals
4. Implement digital versions of the filters
5. Add oversampler/downsampler (aliasing rejection)

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Possible generalization to **Partial Differential Equations** (PDE):

Same principle (cf. Applications)

Part D: Physical/Audio Applications

APPPLICATIONS ON PDEs:

D1. **Nonlinear propagation of a travelling wave (brassy effect)**

switch to powerpoint [Hélie,Smet: IEEE-MED'2008]

D2. **A damped nonlinear string**

results extracted from [Hélie,Roze: JSV'2008]

(not presented here: Moog Ladder Filter
[Hélie: DAFx'2006 & TASLP'2010])

D1. Application 1

NONLINEAR PROPAGATION OF A TRAVELLING WAVE (brassy effect)

- switch to powerpoint [Hélie,Smet: IEEE-MED'2008]

D2. Application 2

A DAMPED NONLINEAR STRING

- Results extracted from [Hélie,Roze: JSV'2008]

D2.1. Model

The Kirchhoff equation (u: transverse displacement)

$$\forall (x,t) \in \Omega =]0;1[\times \mathbb{R}^{+*}$$

$$\frac{\partial^2 u}{\partial t^2} + \quad = \quad 1 \quad \frac{\partial^2 u}{\partial x^2} +$$

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(α, β): damping

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$f(t)$: excitation force

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Boundary and initial conditions

Dirichlet homegeneous:	$u(x=0,t) = u(x=1,t) = 0$
At rest for $t \leq 0$:	$u(x,t) = \partial_t u(x,t) = 0$

D2.2. Equation satisfied by the Volterra kernels

Kirchhoff equation of the string

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \beta \frac{\partial}{\partial t} \frac{\partial^2 u}{\partial x^2} - \left[1 + \varepsilon \int_0^1 \| \frac{\partial u}{\partial x} \|^2 \mathrm{d}x \right] \frac{\partial^2 u}{\partial x^2} - \phi(x) f(t) = 0$$

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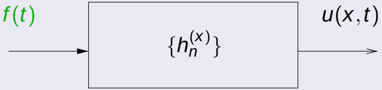
Definition of the solution as a Volterra series



Volterra kernels must be paremetrized in space: $\{h_n\} \rightarrow \{h_n^{(x)}\}$.

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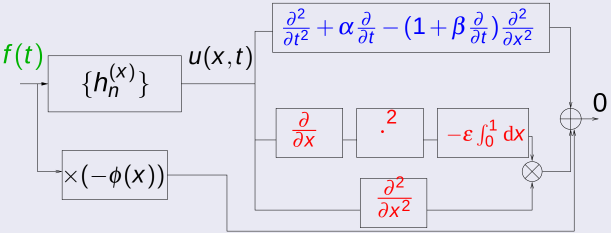
Cancelling system in the time domain



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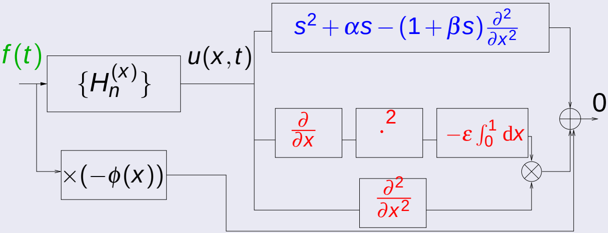
Cancelling system in the Laplace domain



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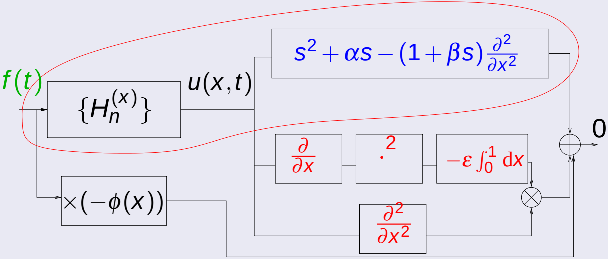
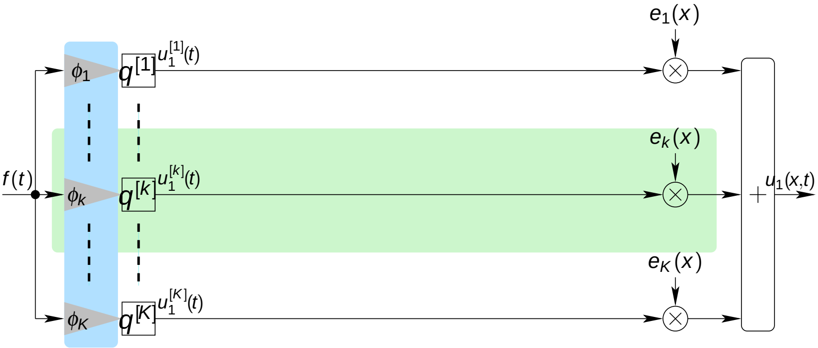


$$\left[(\widehat{s_{1:n}})^2 + \alpha(\widehat{s_{1:n}}) - (1 + \beta(\widehat{s_{1:n}})) \frac{\partial^2}{\partial x^2} \right] H_n^{(x)}(s_{1:n}) \text{ .... etc}$$

D2.3. Solution and realization (details in [JSV 2008])

(Projection of Volterra kernels on the L^2-modal basis $\mathscr{B} = \{e_k(x) = \sqrt{2}\sin(k\pi x)\}$)



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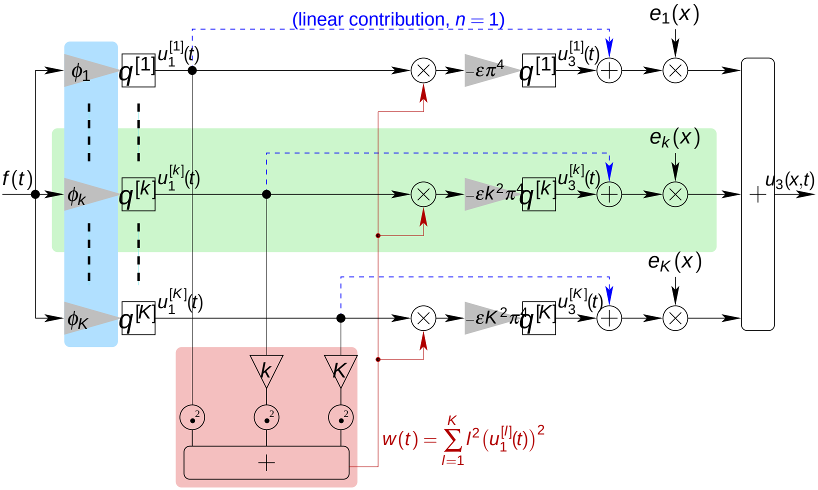

Part E: Convergence of a Volterra series

VOLTERRA SERIES:

Computable convergence domains

E1. A known result(1/2)

(see e.g. [Boyd,1984])

RECALL: definition of a Volterra series

$$x(t) = \sum_{n=1}^{+\infty} x_n(t) \text{ with } x_n(t) = \int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n) u(t-\tau_1) \ldots u(t-\tau_n) \, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n$$

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Bounded Input Bounded Output (BIBO) result ($\|u\|_\infty = \sup_{t\in\mathbb{R}} |u(t)|$)

$$|x_n(t)| = \left| \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n)\, u(t-\tau_1) \dots u(t-\tau_n)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n \right|$$

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Hence, $$\|x\|_\infty \leq \sum_{n=1}^{+\infty} \|x_n\|_\infty \leq \sum_{n=1}^{+\infty} \|h_n\|_1 \left(\|u\|_\infty\right)^n.$$

E2. A known result (2/2) (see e.g. [Boyd,1984])

Gain bound function φ

Define $\varphi(z) = \sum_{n\geq 1} \|h_n\|_1 z^n$ with convergence radius ρ at $z = 0$.

Theorem (BIBO result)

If $\|u\|_\infty < \rho$, **then** the Volterra series expansion of x is normally convergent and

$$\|x\|_\infty \leq \varphi(\|u\|_\infty) < +\infty.$$

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QUESTION

Can we use these theoretical results in practice?

E3. Example: $\dot{x} + ax - \varepsilon x^3 = u$ $a > 0,\ \varepsilon > 0$

Using interconnection laws in the Laplace domain:

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Using interconnection laws in the Laplace domain:

$$(\widehat{s_{1:n}} + a) H_n(s_{1:n})$$

E3. Example: $\dot{x} + ax - \varepsilon x^3 = u$ $\quad a > 0,\ \varepsilon > 0$

Using interconnection laws in the Laplace domain:

$$(\widehat{s_{1:n}} + a) H_n(s_{1:n}) - \varepsilon \sum_{q_1+q_2+q_3=n} H_{q_1}(s_{1:q_1}) H_{q_2}(s_{q_1+1:q_1+q_2}) H_{q_3}(s_{q_1+q_2+1:n})$$

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$$\begin{aligned} H_1(s_1) &= (s_1 + a)^{-1} \quad \text{(one-pole filter)}, \\ H_2(s_{1:2}) &= 0, \end{aligned}$$

E3. Example: $\dot{x} + ax - \varepsilon x^3 = u$ $\quad a > 0,\ \varepsilon > 0$

Using interconnection laws in the Laplace domain:

$$(\widehat{s_{1:n}} + a) H_n(s_{1:n}) - \varepsilon \sum_{q_1+q_2+q_3=n} H_{q_1}(s_{1:q_1}) H_{q_2}(s_{q_1+1:q_1+q_2}) H_{q_3}(s_{q_1+q_2+1:n}) = \delta_{1,n}$$

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Even for this basic example...

Computing $\|h_n\|_1$ is difficult in practice because of the (rapidly) increasing number of terms !

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$x(t) \in \mathbb{X} = \mathbb{R}^p$, A $(p \times p)$, B $(p \times 1)$ and A_m: multi-linear function with norm $\|A_m\|_{\mathscr{ML}} = \sup_{\|X_1\|_{\mathbb{X}}=\dots=\|X_m\|_{\mathbb{X}}=1} \big\|A_m(X_1,\dots,X_m)\big\|_{\mathbb{X}}$.

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Example ($\dot{x} + ax - \varepsilon x^3 = u$, $\quad a > 0$, $\varepsilon > 0$: resistor & NL capacitor)

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Definition (function F)

Let $\kappa = \int_0^{+\infty} \|e^{At}\|_{\mathscr{L}(\mathbb{X},\mathbb{X})} dt$ be the $L^1(\mathbb{R}_+)$-norm of $t \mapsto e^{At}$

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(iv) If $\|u\|_\infty < \rho^\star$, then $\|x\|_\infty \leq \varphi\big(\|u\|_\infty\big) \leq \Psi\big(\|u\|_\infty\big) < +\infty$.

E6. Sketch of proof

- Step 1: We prove that $\|h_n\|_1 \leq \psi_n$ where $\psi_1 = \|h_1\|_1$ and

$$\text{for all } n \geq 2, \quad \psi_n = \kappa \sum_{m=2}^{M} \|A_m\|_{\mathscr{ML}} \sum_{\substack{q_1+\dots+q_m=n \\ q_{1,\dots,m}\geq 1}} \psi_{q_1} \cdots \psi_{q_m}$$

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- Step 3: The singular inversion theorem yields the result.

  (see [Analytic combinatorics, Flajolet & Sedgewick, 2009])

E7. RESULT 2: Truncation error bound

Result (denote $V_N x(t) = \sum_{n=1}^{N} x_n(t)$ and $V_N \Psi(z) = \sum_{n=1}^{N} \psi_n z^n$

If $\|u\|_\infty < \rho^\star$, then

$$\|x - V_N x\|_\infty \leq \underbrace{\Psi(\|u\|_\infty) - V_N \Psi(\|u\|_\infty)}_{\text{Remainder of } \psi} \leq \sigma \frac{\left(\|u\|_\infty / \rho^\star\right)^{N+1}}{1 - \|u\|_\infty / \rho^\star} .$$

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Sketch of proof:

- Step 1: We prove that $\psi_n \leq \sigma / (\rho^\star)^n$ using Cauchy etimates.
- Step 2: We prove that if $z \in \mathbb{C}$, $|z| < \rho^\star$, then for $N \geq 1$,

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E8. Back to the example

Example ($\dot{x} + ax - \varepsilon x^3 = u, \qquad a > 0,\ \varepsilon > 0$)

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$h_1(t) = \mathrm{e}^{-at}$ on $\mathbb{R}_+$ and $\|h_1\|_1 = \kappa = 1/a$ so that

$$F(X) = 1/(a - \varepsilon X^2).$$

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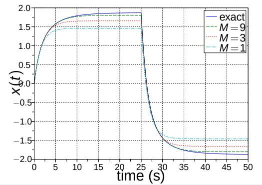
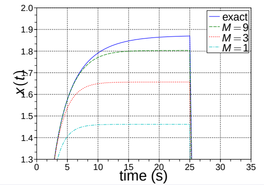
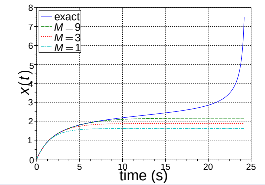
Remark

When there is no closed-form solution for $\|h_1\|_1$, κ, σ and $\rho^\star$, equations can be numerically solved.

E9. Numerical simulations ($a = 0.65$ and ε s.t. $\rho^\star = 1$)

Square input: $u(t) = e$ on $[0, 25)$ and $u(t) = -e$ on $[25, 50)$, etc.

$e = 0.95 < \rho^\star$	Idem: ZOOM	$e = 1.05 > \rho^\star$
		

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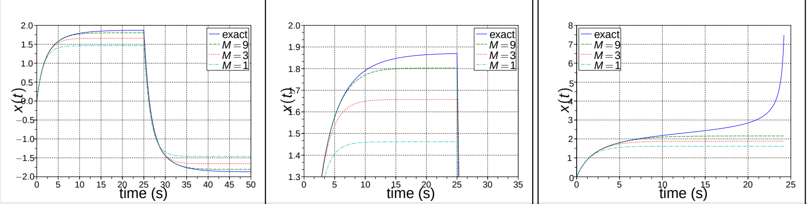

- $e < \rho^\star$: the VS converges to the trajectory of the NL system
- $e > \rho^\star$: the VS becomes divergent and the trajectory of the nonlinear system gets out of the domain of attraction of 0.

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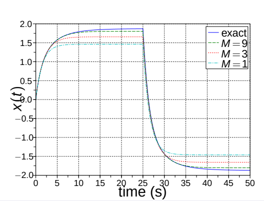

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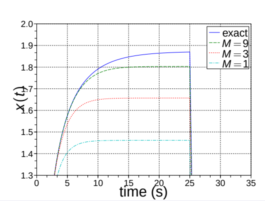

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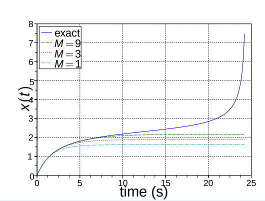

- $e < \rho^\star$: the VS converges to the trajectory of the NL system
- $e > \rho^\star$: the VS becomes divergent and the trajectory of the nonlinear system gets out of the domain of attraction of 0.

In this example (but not in general), $\rho^\star = \rho$ and the convergence domain coincides with the domain of attraction of 0.

E10. IN SUMMARY: Volterra series convergence

Sufficient condition of convergence: $\|u\|_\infty < \rho^\star$

- Consider system $\dot{x} = Ax + Bu + \sum_{m=2}^{M} A_m(x, \ldots, x)$, with $x(0) = 0$.
- Compute $\kappa = \|e^{At}\|_1$, $\|h_1\|_1 = \|e^{At}B\|_1$, and $\|A_m\|_{\mathscr{M}\mathscr{L}}$.
- Introduce $F(X) = \|h_1\|_1 / \left(1 - \kappa \sum_{m=2}^{M} \|A_m\|_{\mathscr{M}\mathscr{L}} X^{m-1}\right)$.
- Solve $F(\sigma) - \sigma F'(\sigma) = 0$ for $\sigma > 0$.
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- Compute the guaranted bound $\rho^\star = \sigma / F(\sigma) \quad (\leq \rho)$.

Some generalizations are available for:

- Non zero initial conditions
- Analytic nonlinearities w.r.t. x and affine w.r.t. u
- Multiple inputs $u(t) \in \mathbb{R}^q$
- Some nonlinear PDEs

Part F: Conlusion

CONCLUSION

F1. Conclusion

In this tutorial:

- Volterra series are used to represent, analyze and simulate some Input/Output systems which include distortions
- **Derivation of the kernels** for a given system (ODE or PDE)
- **Audio applications** and **simulations** based on linear filters, sums and products
- **Computable convergence radius** and guaranteed truncation error bound in some cases

F2. Conclusion

Other works and (examples of) perpectives

- **Identification** of systems
  For Hammerstein models, see e.g. [Farina: AES'108, 2000], [Novak et al.: IEEE-TIM, 2010] and [Rébillat et al.: JSV, 2011]
- **Model order reduction** based on Volterra kernels
- Convergence criterion for **analytic systems** (w.r.t. x and u)
- Convergence criterion for a **large class of PDEs**

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