Volterra series: Identification problems and nonlinear order separation

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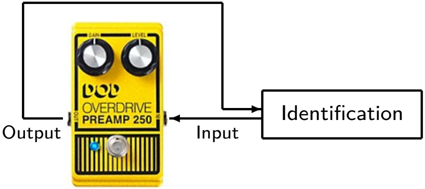
Context and motivations

Thesis (started in October 2015)

- Goal: Realist instrumental sound synthesis (tone evolution VS. note and loudness)
- Idea:



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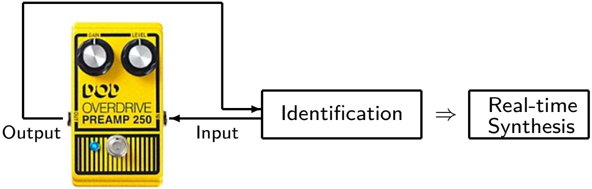

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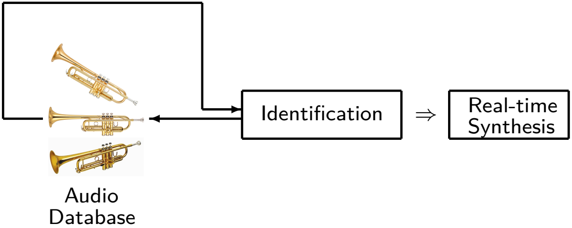

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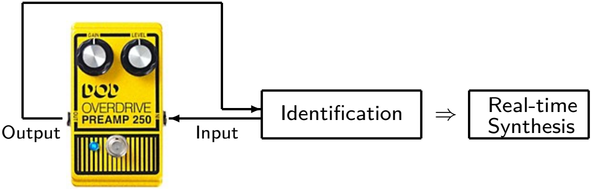
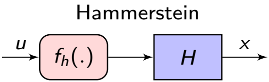


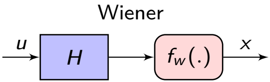
Nonlinear systems under study:

Distortion pedal, compressor, loudspeaker, nonlinear resonator, ...

- Fading memory
- Regular nonlinearities:
  - ➢ Taylor-like expansion
  - ➢ No chaotic behaviour

Nonlinear system representation





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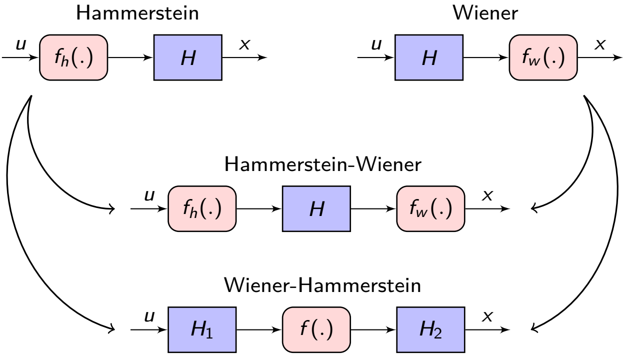

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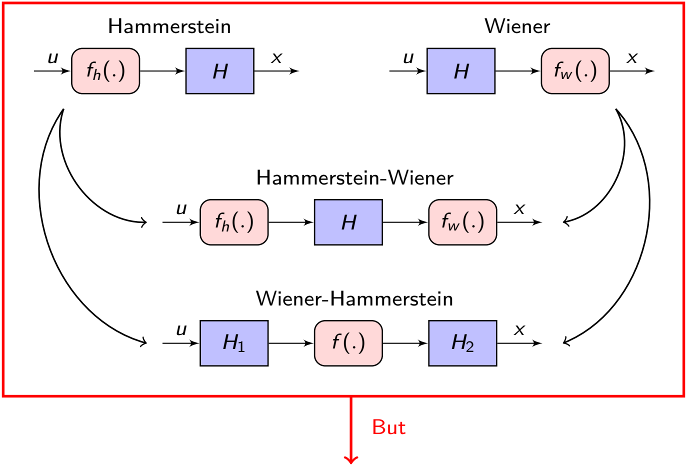
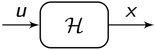


Description not adapted to physical systems

Volterra series



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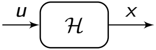

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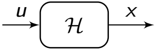

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Remarks

- General representation for regular nonlinearities
- ∃ kernel representation in the spectral domain
- Interconnection laws (sum, product, cascade) in temporal & spectral domain
- Representation only valid:
  - ➢ around an equilibirum (here $x_0 = 0$)
  - ➢ in a computable convergence domain [Hélie and Laroche, 2011]
- Cannot perform hysteresis or chaotic behaviour

Summary

From linear to nonlinear systems

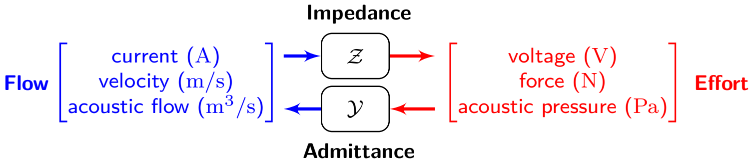
- Immitance inversion (impedance $\leftrightarrow$ admittance)
- Passivity
- System identification

Nonlinear order separation

- Order homogeneity in Volterra
- Idea 1: Using amplitude [Boyd et al., 1983]
- Idea 2: Using phase
- Simulation results

Immitance inversion (impedance $\leftrightarrow$ admittance)

Notion of *immitance*



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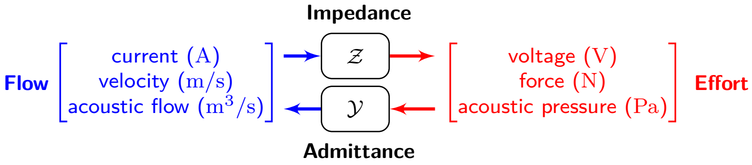

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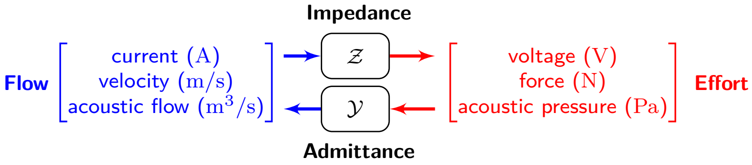

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$$Y_1 \equiv \frac{1}{Z_1} \quad \text{and} \quad Y_n \equiv \text{fct}(Z_1, \ldots, Z_n) \quad \text{for } n \geq 2$$

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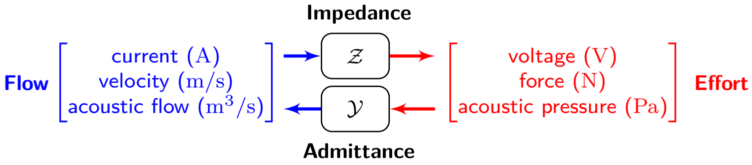

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Conclusion: $\exists$ well-defined inversion for nonlinear systems

Passivity ($u \to$ Immitance $\to x$)

Definition (Passivity [Youla et al., 1959; Boyd and Chua, 1982])

A system is *passive* if and only if it does not return more energy than it consumed, i.e.:

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$u[k] \to \sum_{l=0}^{L} h_n[l] u[k-l]^n$	discrete time volterra kernel homogeneous order	Positivity of the eigenvalues of supersymmetric tensor $\mathbb{F}_n$ associated with h_n [Qi, 2005]

In general: Open problem, no solution yet (work in progress)

System identification

Idea: Computing the system *function* from input and output.

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Linear systems

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Nonlinear systems

⚠ Notion of *transfer function* not valid

- For Volterra series: theoretical work by Boyd et al. [1983, 1984]
  Order 1 & 2 in practice, robustness problems
- For Hammerstein system: Farina [2000]; Rébillat et al. [2011]; Novák et al. [2010]
  Method robust, efficient and quick

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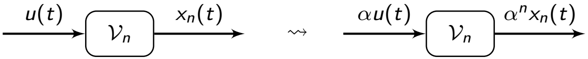
Multilinearity of Volterra kernels



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Advantages and disadvantages

- ✔ easy to implement
- ✖ amplitude spanning a large range $\rightsquigarrow$ **measurement error**
- ✖ matrix ill-conditioned $\rightsquigarrow$ **numerical error**

Idea 2: Using phase

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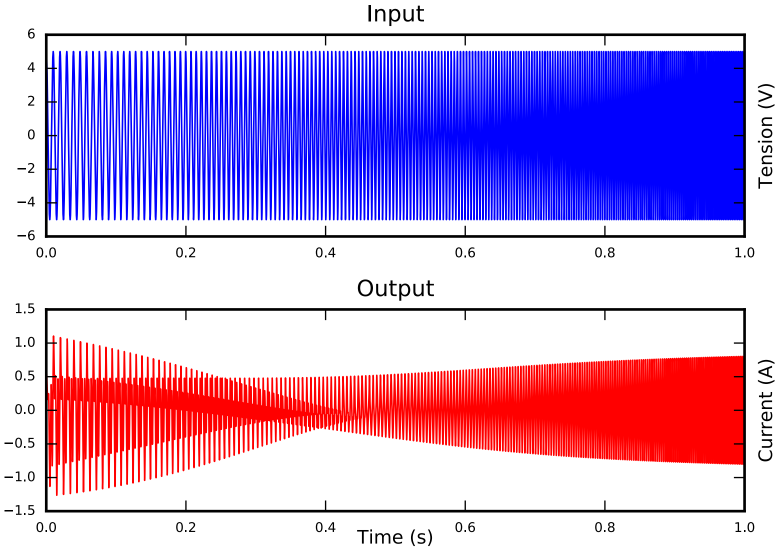
Advantages and disadvantages

- ✔ only one amplitude ⇝ **less measurement error**
- ✔ DFT matrix is well-conditioned
- ✔ Can use FFT ⇝ **numerical computation**
- ✖ Need for complex signals ⇝ **only theoretical**

Simulation results (1)



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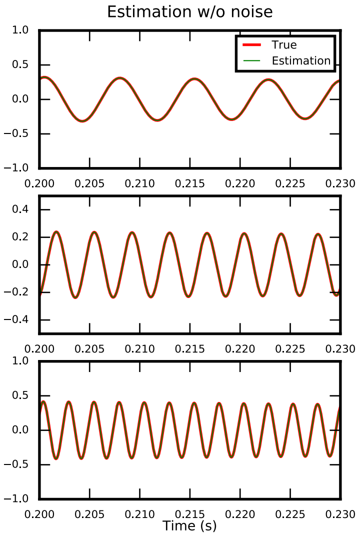

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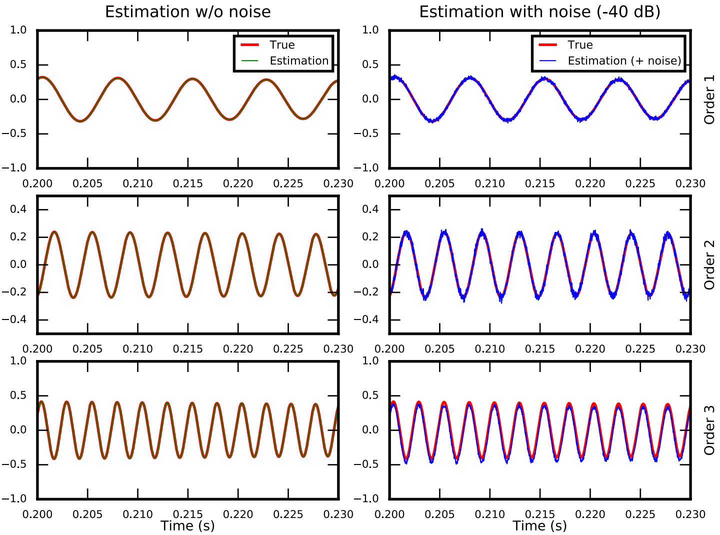

Conclusion

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... with still a lot of open questions

- Still no criterion for system passivity (work in progress)
- No general and robust method for identification
- Nonlinear order separation is still not resolved:
  adaptation of the "phase" method for real input (work in progress)

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